Paper to ISDE5
Apr. 2007

Arithmetic and Efficient Fourier Transform for a Uniform and Multiresolutional Digital Earth Model[*]

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Keywords: Fourier Transform, Digital Earth Reference Model, Hexagonal Grid, Multiresolution

0. Introduction

Traditional image processing algorithms and digital image transforms are usually carried out on pixels of square grids. However the pixels of hexagonal grids provide for higher packing density of discs and give a more accurate approximation of circular regions than that of square grids. Furthermore the pixels of hexagonal grids are uniformly connected in the sense that the distance from a given pixel to any adjacent pixel is the same. The set consisting of all centers of pixels of a hexagonal grid is called a *hexagonal lattice* and a nonempty subset of a hexagonal lattice is called a *hexagonal array*. In general, a *2-dimensional lattice* is the set of all integer linear combinations of two independent vectors of a 2-dimensional vector space R^2, where R denotes the set of real numbers. The elements of a lattice are called *lattice points*. The *Voronoi cell* of a lattice point x in a 2-dimensional lattice L is a set consisting of all points of R^2 which are at least as close to x as to any other lattice point of L.

The *Pyxis structure*, denoted P, consists of a sequence of hexagonal arrays together with an elegant method for labeling the lattice points. It was proposed by Peterson [Peterson, 2003] to provide a model for uniquely referencing all the sampled points of the earth's surface which is sampled uniformly and in multiresolutions. The n^{th} array in the sequence of the Pyxis structure is called its n^{th} *level* and denoted $P(n)$. Figure 1 shows the Voronoi cells of the first four levels of the Pyxis structure. Consider a sphere which is tessellated by 20 regular hexagons and 12 regular pentagons. More precisely, each of those 32 polygons should be spherical, i.e., a part of the sphere. Figure 2 shows the sphere with those 32 polygons flattened onto the plane, and Figure 3 displays each pentagon in Figure 2 as a hexagon with one of its 6 directions empty. For each side of those polygons in Figure 3, make a line segment with length being equal to that of the given side divided by $\sqrt{3}$ such that this line segment and the given side are perpendicular to and bisect each other. Then we obtain Figure 4 which in turn follows Figure 5. Figures 2, 4, and 5 correspond to the *division of the sphere at level 0, 1 and 2,* respectively. Continue such division, we can obtain the *division of the sphere at level* n for any integer n. As shown in Figure 5, for any integer $n > 1$, the set of all sampled points of the sphere at the level n is a disjoint union of 20 copies of $P(n-1)$ and 12 copies of $P(n)$ by omitting one of its 6 directions.

This research deals with the arithmetic and discrete Fourier transform (DFT) for the Pyxis structure. The addition of Pyxis labels corresponding to the vector addition of lattice points such as 0506+2005=1040 in Figure 1 is important for quick data retrieval and the DFT. We search for efficient algorithms to add any two labels of the Pyxis structure and those algorithms are shown

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[*] Supported by the Pyxis Innovation Inc., Kingston, Ontario, Canada

to be of linear computational complexity of n for adding any two labels of $P(n)$. The DFT is one of the important tools in image processing. We show that, for any integer $n > 2$, the DFT on the n^{th} level of the Pyxis structure cannot be formulated using the method in Section 3 by way of a set of coset representatives of the quotient group of two lattices. However we can modify any level of the Pyxis structure slightly by adding points of the neighboring arrays to obtain a hexagonal shaped array whose DFT can be computed efficiently.

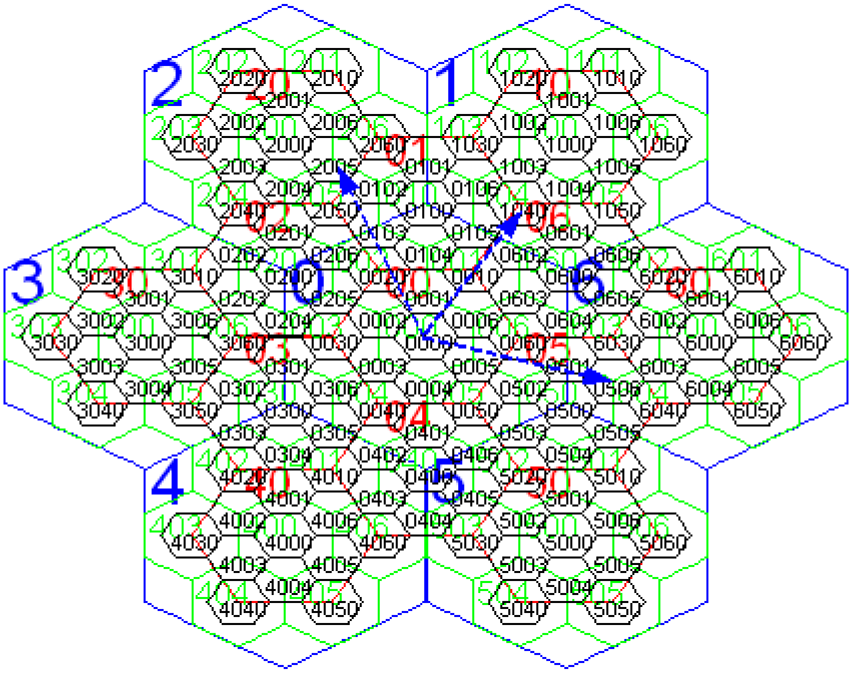

Figure 1 - The first four levels of the Pyxis structures, where the 3 dashed vectors show the addition of two labels, 0506+2005=1040

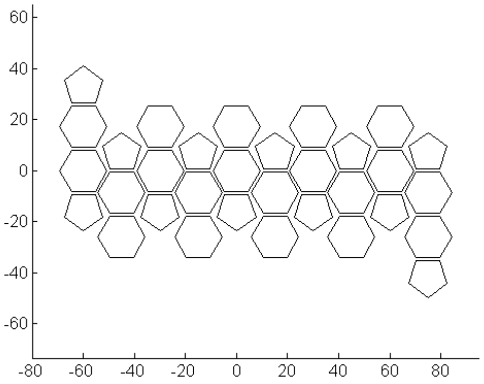

Figure 2 - The 20 hexagons and 12 pentagons used to tessellate the sphere

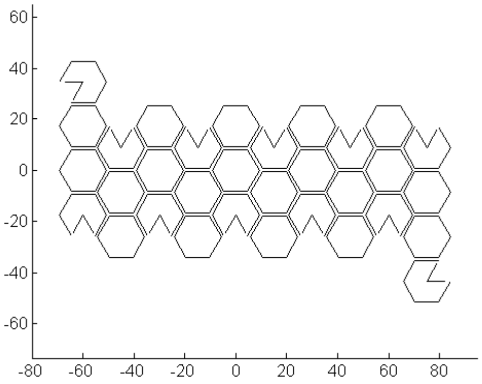

Figure 3 - Display each pentagon in Figure 2 as a hexagon with one of its 6 directions empty

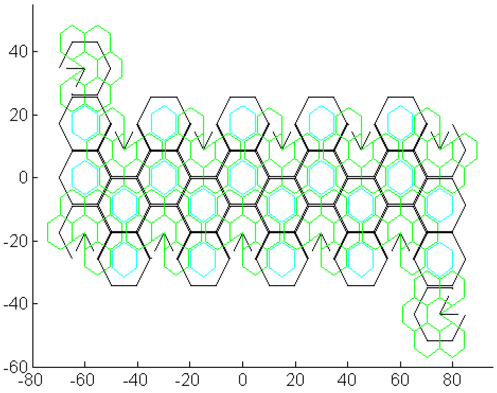

Figure 4 - The green and blue polygons obtained from the division of polygons from Figure 3

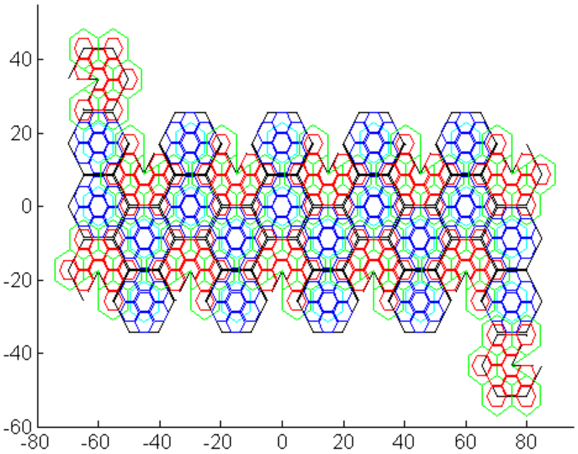

Figure 5 - The red and blue hexagons are generated from the division of ploygons in Figure 4

1. Algebraic definition, labeling, and addition of labels of the Pyxis structure

Throughout this paper, N, Z, R, and C denote the set of positive integers, integers, real numbers, and complex numbers, respectively. To consider the DFT on the Pyxis structure, we need its algebraic definition. If v_1 and v_2 are two linearly independent vectors in R^2, then the set defined by $L = \{n_1 v_1 + n_2 v_2 \mid n_1, n_2 \in N\}$ is called a *2-dimensional lattice,* and $\{v_1, v_2\}$ is called *a set of generators* of L. In the following, let $V_1^A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $V_2^A = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$, $V_1^B = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$, $V_2^B = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$. Let ρ be a fixed positive number and, for any integer $n \geq 1$ and $k = 1,2$, let

$$V_{2n-1,k} = \left(\frac{1}{\sqrt{3}}\right)^{2n-1} \rho V_k^A, \quad V_{2n,k} = \left(\frac{1}{\sqrt{3}}\right)^{2n} \rho V_k^B.$$

For any $n \in N$, let $L_n = \{n_1 V_{n,1} + n_2 V_{n,2} \mid n_{,}, n_2 \in Z\}$. Obviously $V_{2n-1,1} = V_{2n,1} - V_{2n,2}$ and $V_{2n-1,2} = V_{2n,1} + V_{2n,2}$. Hence we have $L_{2n-1} \subseteq L_{2n}$. Similarly $L_{2n} \subseteq L_{2n+1}$.

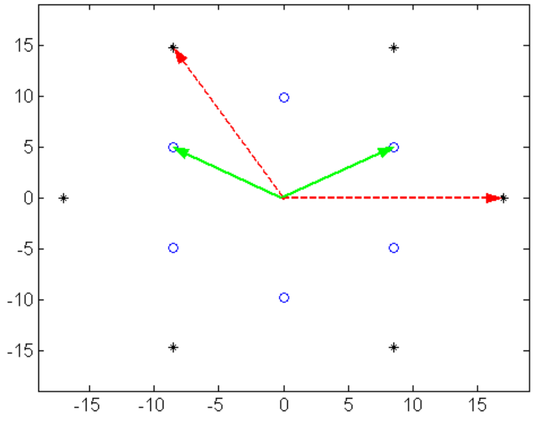

Figure 6 - The two dashed red vectors are V , , the green vectors are , , consists of the six * points, and consists of the six o pints

To give an algebraic definition of the Pyxis structure, for any $n \in N$, we let $\beta_n = \{n_1 V_{n,1} + n_2 V_{n,2} \mid (n_1, n_2) \in D\}$, where $D = \{(1,1), (0,1), (-1,0), (-1,-1), (0,-1), (1,0)\}$. Also let

$\overline{\beta}_n = \beta_n \cup \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$. Figure 6 shows the generators of the lattices L_1 and L_2 as well as the elements of β_1 and β_2. In the following, for any lattice L and $\phi \neq X, Y \subseteq L$, let $X + Y = \{x + y \mid x \in X, y \in Y\}$.

Definition 1 Let $P(0)$ be the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $P(1)$ consist of the seven lattice points of $\overline{\beta}_1$, and for any $1 < n \in N$ let $P(n) = P(n-1) \bigcup \{P(n-2) + \beta_n\}$. The set $P(n)$ is called *the Pyxis structure at level n* for any $0 \leq n \in Z$.

We can prove that $P(n) \subset L_n$ for any $n > 0$. Furthermore we have the following unique representation Theorem for the lattice points of $P(n)$ whose proof is provided in [Zheng , 2007] and omitted here, where $\overline{D} = D \cup \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$.

Theorem 1 For any $0 \leq n \in N$, there exist uniquely determined pairs $(q_{k,1}, q_{k,2}) \in \overline{D}$ for $k = 1,2,...,n$ such that $Q = \sum_{k=1}^{n} q_{k,1} V_{k,1} + q_{k,2} V_{k,2}$, where $(q_{k,1}, q_{k,2}) = (0,0)$ or $(q_{k+1,1}, q_{k+1,2}) = (0,0)$ for any $1 \leq k < n$.

The expression in Theorem 1 is called the *standard expression* of $Q \in P(n)$. Based on this unique representation, now we can label the lattice points of $P(n)$ as follows.

Labeling rule of the Pyxis structure: Let $D_0 = (0,0)$, $D_1 = (1,1)$, $D_2 = (0,1)$, $D_3 = (-1,0)$, $D_4 = (-1,-1)$, $D_5 = (0,-1)$, and $D_6 = (1,0)$. Then $\overline{D} = \{D_j \mid j = 0,1,2,...,6\}$. Hence in Theorem 1, for any $n \in N$, $Q \in P(n)$, and $1 \leq k \leq n$, there exists a uniquely determined $Q_k \in \{0,1,2,...,6\}$ such that $(q_{k,1}, q_{k,2}) = D_{Q_k}$. The string $Q_1 Q_2 ... Q_n$ is called *the label of Q in Λ_n*, where $\Lambda_n = \{X_2 X_2 ... X_n \mid X_i = 0,1,2,...,6, \forall i; X_i \bullet X_{i+1} = 0, \forall i < n\}$. It is easy to verify that the center hexagon of $P(1)$ is labeled 0 and other 6 hexagons of $P(1)$ are labeled 1,2,3,4,5,6 going counter-clockwise. $P(2)$ consists of 13 hexagons labeled 00, 10, 20, 30, 40, 50, 60, 01, 02, 03, 04, 05 and 06. Because there is a one to one correspondence between Voronoi cells and lattice points of a lattice, we use the same label for both a lattice point and its corresponding Voronoi cell.

If $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq Q \in P(1)$, then $Q + Q \notin P(1)$. However, $\frac{Q}{3} \in P(3)$ for any $Q \in P(1)$ since $V_{3,j} = \frac{1}{3} V_{1,j}$ for $j = 1,2$, and it is easy to show that $\frac{Q}{3} + \frac{R}{3} \in P(3)$ for any $Q, R \in P(1)$. For any $Q \in P(n)$, let $\tau(Q) = \frac{Q}{3}$ and call $\tau(Q)$ the *compression of Q by a factor of 3*. Because $V_{n+2,j} = \frac{1}{3} V_{n,j}$ for $j = 1,2$, we have $\tau(Q) \in P(n+2)$. Similar to the case of $P(1)$, we can prove the following two lemmas.

Lemma 2 For any $Q \in P(n)$, the label of Q in Λ_n is λ if and only if the label of $\tau(Q)$ in Λ_{n+2} is 00λ.

Lemma 3 For any $Q, R \in P(n)$, we have $\tau(Q) + \tau(R) \in P(n+2)$.

Using the unique representation in Theorem 1, we have the following proposition which tells whether or not $Q + R \in P(n)$ based on the label of $\tau(Q) + \tau(R)$. Its proof is omitted here.

Proposition 4 For any $Q, R \in P(n)$, let $W = \tau(Q) + \tau(R)$ and let $W_1 W_2 W_3 ... W_{n+2} \in \Lambda_{n+2}$ be the label of W in Λ_{n+2}. Then $Q + R \in P(n)$ if and only if $W_1 = W_2 = 0$.

Addition of the labels of the Pyxis structure: Let $n \in N$ and $Q, R \in P(n)$, and let $Q_1 Q_2 ... Q_n$ and $R_1 R_2 ... R_n$ be the labels of Q and R in Λ_n, respectively. We consider the following two cases.

Case 1: If $Q + R \in P(n)$, then the label of $Q + R$ in Λ_n is called the *sum of the labels* $Q_1 Q_2 ... Q_n$ *and* $R_1 R_2 ... R_n$. Also write $Q_1 Q_2 ... Q_n \oplus R_1 R_2 ... R_n = S_1 S_2 ... S_n$ in Λ_n where $S_1 S_2 ... S_n$ is the label of $Q + R$ in Λ_n.

Case 2: If $Q + R \notin P(n)$, then we consider the label of $\tau(Q) + \tau(R)$ since, by Lemma 3, $\tau(Q) + \tau(R) \in P(n+2)$ and since $\tau(Q)$ and $\tau(R)$ are just the compression of Q and R by a factor 3 respectively. Also, by Lemma 2, the label of $\tau(Q)$ is $00Q_1 Q_2 ... Q_n$ and the label of $\tau(R)$ is $00R_1 R_2 ... R_n$. If $00R_1 R_2 ... R_n \oplus 00Q_1 Q_2 ... Q_n = S_1 S_2 S_3 S_4 ... S_{n+2}$ in Λ_{n+2}, then $S_1 S_2$ and $S_3 S_4 ... S_n S_{n+1} S_{n+2}$ are called the *carry* and the *remainder of the addition* $Q_1 Q_2 ... Q_n \oplus R_1 R_2 ... R_n$, respectively.

In Case 1, we have $00Q_1 Q_2 ... Q_n \oplus 00R_1 R_2 ... R_n = 00S_1 S_2 ... S_n$ in Λ_{n+2}. Hence 00 and $S_1 S_2 ... S_n$ are called the *carry* and the *remainder of the addition* $Q_1 Q_2 ... Q_n \oplus R_1 R_2 ... R_n$, respectively.

The following Table 1 and Table 2 are partial addition tables for Λ_3 and Λ_4, respectively, and will be applied in the algorithms for the addition of two labels in Λ_n.

Table 1 - Sum of a label of the form $00i$ and a label of the form $00i$ or $0i0$ in Λ_3

$\oplus$	**001**	**002**	**003**	**004**	**005**	**006**	**010**	**020**	**030**	**040**	**050**	**060**
001	104	010	002	000	006	060	103	205	003	005	603	105
002	010	205	020	003	000	001	206	204	306	004	006	104
003	002	020	306	030	004	000	205	301	305	401	005	001
004	000	003	030	401	040	005	002	306	402	406	502	006
005	006	000	004	040	502	050	001	003	401	503	501	603
006	060	001	000	005	050	603	104	002	004	502	604	602

Table 2 - Sum of a label of the form $000i$ and a label of the form $000i$ or $00i0$ in Λ_4

$\oplus$	**0001**	**0002**	**0003**	**0004**	**0005**	**0006**	**0010**	**0020**	**0030**	**0040**	**0050**	**0060**
0001	0104	0020	0002	0000	0006	0010	0105	0103	0205	0003	0005	0603
0002	0020	0205	0030	0003	0000	0001	0104	0206	0204	0306	0004	0006
0003	0002	0030	0306	0040	0004	0000	0001	0205	0301	0305	0401	0005
0004	0000	0003	0040	0401	0050	0005	0006	0002	0306	0402	0406	0502
0005	0006	0000	0004	0050	0502	0060	0603	0001	0003	0401	0503	0501
0006	0010	0001	0000	0005	0060	0603	0602	0104	0002	0004	0502	0604

2. Algorithms for the addition of the labels of $P(n)$

In this section, we develop algorithms to add labels of $P(n)$. By the definition of labels of $P(n)$ and the definition of the addition of labels in Λ_n, we have the following lemma which will be useful for the addition algorithms of Λ_n.

Lemma 5 For any $n \in N$ and $1 \le k \le n$, we have the following.

(1). If $\lambda = \lambda_1\lambda_2...\lambda_n \in \Lambda_n$, then we have $\lambda = \lambda_1\lambda_2...\lambda_k 00...0 \oplus 00...0\lambda_{k+1}\lambda_{k+2}...\lambda_n$ in Λ_n.

(2). If $\lambda_1\lambda_2...\lambda_k 00...0 \in \Lambda_n$, $00...0\lambda_{k+1}\lambda_{k+2}...\lambda_n \in \Lambda_n$, and either $\lambda_k = 0$ or $\lambda_{k+1} = 0$, then $\lambda_1\lambda_2...\lambda_k 00...0 \oplus 00...0\lambda_{k+1}\lambda_{k+2}...\lambda_n = \lambda_1\lambda_2...\lambda_k\lambda_{k+1}\lambda_{k+2}...\lambda_n$ in Λ_n.

(3). For any $A = A_1A_2...A_n \in \Lambda_n$ and $B = B_1B_2...B_n \in \Lambda_n$, if $A \oplus B = C_1C_2...C_n \in \Lambda_n$, then, in Λ_{n+1}, $A_1A_2...A_n0 \oplus B_1B_2...B_n0 = C_1C_2...C_n0$ and $0A_1A_2...A_n \oplus 0B_1B_2...B_n = 0C_1C_2...C_n$.

Let us take $n = 6$ for example. The other cases can be done similarly. Let $A = A_1A_2A_3A_4A_5A_6 \in \Lambda_6$ and $B = 0000B_5B_6 \in \Lambda_6$. By Lemma 5 (1), we have $A = A_1A_2 0000 \oplus 00A_3A_4 00 \oplus 0000A_5A_6$ in Λ_6. Also by Table 2 we have $A_5A_6 \oplus B_5B_6 = C_1C_2C_3C_4$ in Λ_4 for some $C_1C_2C_3C_4 \in \Lambda_4$. It follows that $0000A_5A_6 \oplus 0000B_5B_6 = 00C_1C_2C_3C_4 \in \Lambda_6$. Hence

$$A \oplus B = A_1A_2 0000 \oplus 00A_3A_4 00 \oplus 00C_1C_2 00 \oplus 0000C_3C_4 \text{ in } \Lambda_6. \qquad \textbf{(1)}$$

If either $A_4 = 0$ or $C_3 = 0$, then, by Lemma 5 (1), C_3C_4 is the last two digits of $A \oplus B$, and Equation (1) and Lemma 5 (3) follow that the other digits of $A \oplus B$ can be obtained from $A_1A_2 00 \oplus 00A_3A_4 \oplus 00C_1C_2$. Otherwise we have $A_4 \ne 0$ and $C_3 \ne 0$ and hence, by Theorem 1, we have $A_3 = A_5 = 0$ and $C_2 = C_4 = 0$. Using Table 1, we have $0C_3 \oplus A_4 0 = D_1 0 D_3$ in Λ_3 for some $D_1 0 D_3 \in \Lambda_3$. It follows that $00C_3C_4 \oplus A_3A_4 00 = D_1 0 D_3 0$ in Λ_4. Hence Equation (1) becomes the following Equation.

$$A \oplus B = A_1A_2 0000 \oplus 00C_1 000 \oplus 00D_1 0 D_3 0 \text{ in } \Lambda_6. \qquad \textbf{(2)}$$

By Lemma 5 (1), it follows from Equation (2) that the last two digits of $A \oplus B$ is $D_3 0$ and the other digits of $A \oplus B$ can be obtained from $A_1A_2 00 \oplus 00C_1 0 \oplus 00D_1 0$.

We have just shown how to determine the last two digits of the sum of $A = A_1A_2A_3A_4A_5A_6$ and $B = 0000B_5B_6$. If the first four digits of B are not all zeros, then there are more cases to be considered and there are more additions. To obtain other digits of the sum of A and B, the computation seems to be much more. The computational complexity of adding labels $A = A_1A_2...A_n$ and $B = B_1B_2...B_n$ seems to be at least of order $n + (n-1) + (n-2) + ... + 1 = \frac{n(n+1)}{2}$, i.e., of order n^2. However, in the following, we will show algorithms which have computational complexity of order n. The following Algorithm 1 utilizes Lemma 3. Assume n is even. To add $A = A_1A_2...A_n$ and $B = B_1B_2...B_n$, as shown in the paragraph preceding Equation 1, we first add $A_{n-1}A_n$ and $B_{n-1}B_n$ to get $C_1C_2C_3C_4$. If $C_3 = 0$, then the last two digits of the sum $A \oplus B$ is C_3C_4. Otherwise, if $A_{n-1} \ne 0$ or $B_{n-1} \ne 0$, *using* *table 1*, add C_2C_3 to $A_{n-2}A_{n-1}$ or $B_{n-2}B_{n-1}$, or add the three terms C_2C_3, $A_{n-2}A_{n-1}$ and $B_{n-2}B_{n-1}$, to get a sum $X_3X_4R_1$. Let $R_2 = 0$. Then R_1R_2 is the last two digits of the sum $A \oplus B$, and X_3X_4 is the carry of adding the last two digits of A and B. Let us call C_1C_2 and C_3C_4 the *temporary carry* and *temporary remainder of adding the last two digits of* A *and* B, respectively, and call X_3X_4 and R_1R_2 the *actual carry* and *actual remainder of adding the last two digits of* A *and* B, respectively. It is easy to see that the computational time from a temporary carry to an actual carry is bounded. By Lemma 3, we can apply a similar idea to obtain other digits in the sum $A \oplus B$. For any $1 \le i < j \le n$, $A(i:j)$ denotes the string $A_iA_{i+1}...A_j$, and $B(i:j)$ denote $B_iB_{i+1}...B_j$. We also use $tempC$ and $temp\Re$ to indicate a temporary carry and a temporary remainder, respectively.

Algorithm 1 Input: An integer $n > 3$ (assume n is even, otherwise adjoin a 0 to the right end of each label and finally delete the 0 adjoined), and two labels $a_1a_2...a_n, b_1b_2...b_n \in \Lambda_n$.

Output: The $SUM = 00a_1a_2...a_n \oplus 00b_1b_2...b_n$ in Λ_{n+2}.

Step 1: Let $M = \frac{n}{2}$ and, for $i = 1$ to M, let $\mathrm{P}_a(i) = A((2i-1):2i)$ and $\mathrm{P}_b(i) = B((2i-1):2i)$.

Step 2: Let SUM be the empty set, and add $P_a(1)$ and $P_b(1)$ to get a $tempC$ and a $temp\Re$.

Step 3: For $i = 2$ to M, do the following.

Let $carrySet_1$ consist of elements of the set $\{P_a(i), P_b(i), tempC\}$ whose first digit is 0, and let $carrySet_2$ consist of other elements of that set. If the first digit of $temp\Re$ is 0, let $remd$ (means remainder) be the $temp\Re$, and add the elements in $carrySet_1$ and $carrySet_2$ to update $tempC$ and $temp\Re$. Else if $carrySet_1$ is non empty, add $temp\Re$ and the elements of $carrySet_1$ to update $remd$ and $carrySet_2$. Then add the elements of $carrySet_2$ to update $temp\Re$ and $tempC$. Else let the $remd$ be the $temp\Re$ and add all elements of $carrySet_2$ together to update $temp\Re$ and $tempC$. Let SUM be the concatenation of $remd$ and SUM, and increase i.

Step 4: Let SUM be the concatenation of the $tempC$, $temp\Re$ and SUM.

When many labels are added simultaneously, the following Algorithm 2 needs less time than Algorithm 1. Algorithm 2 needs the following Table 3 and division by 3 subroutine.

Table 3 – Digit Set

(0,0)	(2,0)	(1,1)	(-1,1)	(-2,0)	(-1,-1)	(1,-1)	(3,1)	(0,2)	(-3,1)	(-3,-1)	(0,-2)	(3,-1)
00	01	02	03	04	05	06	10	20	30	40	50	60

Division by 3 subroutine: Let (a,b) be a pair of integers. Find a', b', r, and s such that $(a,b) = 3(a',b') + (r,s)$, where $(r,s) \in \{(0,0),(1,1),(1,-1),(-1,-1),(-1,1),(2,0),(0,2),(-2,0),(0,-2)\}$. For example $(3,7) = 3(1,3) + (0,-2)$.

Algorithm 2 Input: Integer n (assume n is odd, otherwise adjoin a 0 to the right end of each label and finally delete the 0 adjoined), and a subset $\{N_1, N_2, ..., N_m\} \subseteq \Lambda_n$.

Output: The $SUM = N_1 \oplus N_2 \oplus ... \oplus N_m$ in Pyxis notation.

Step 1: Adjoin a 0 to the left of each of the labels $N_1, N_2, ..., N_m$.

Step 2: Initialize $k = 0$ and, for each of the labels $N_1, N_2, ..., N_m$, convert the rightmost pair of digits to a pair of integers as given by Table 3. Then sum those pairs to get an answer (a_1, b_1). For example if $N_1 = 401$ and $N_2 = 302$, then $(a_1, b_1) = (2,0) + (1,1) = (3,1)$.

Step 3: While $k \le \frac{n}{2}$ or $(a_1, b_1) \ne (0,0)$ do the following a), b) and c).

a) $k \leftarrow k+1$.

b) For each of the labels $N_1, N_2, ..., N_m$, convert the pair of Pyxis labels consisting of $(2k)^{th}$ and $(2k+1)^{th}$ digits (ordered from the rightmost) to a pair of integers as given by Table 3. Then sum those pairs to get an answer (a_2, b_2). For example if $k=1, N_1 = 0401$ and $N_2 = 0302$, then $(a_2, b_2) = (-2,0) + (-1,1) = (-3,1)$.

c) Apply Division by 3 subroutine on (a_1, b_1) to obtain (a_1', b_1') and (r_1, s_1). $(a_1, b_1) \leftarrow (a_2 + a_1', b_2 + b'_1)$. Apply Division by 3 subroutine on (a_1, b_1) to obtain (r_2, s_2). If $s_1 \neq \pm 2$ or if ($s_1 = \pm 2$ and $r_2 = 0$), then $t_k = (r_1, s_1)$ and convert t_k to Pyxis notation using Table 3; else if $s_1 = 2$ and if $(r_2, s_2) \in \{(1,1), (-2,0), (1,-1)\}$, then $t_k = 60$ and $(a_1, b_1) \leftarrow (a_1 - 1, b_1 + 1)$; else if $s_1 = 2$ and if $(r_2, s_2) \in \{(2,0), (-1,-1), (-1,1)\}$, then $t_k = 40$ and $(a_1, b_1) \leftarrow (a_1 + 1, b_1 + 1)$; else if $s_1 = -2$ and if $(r_2, s_2) \in \{(1,1), (-2,0), (1,-1)\}$, then $t_k = 10$ and $(a_1, b_1) \leftarrow (a_1 - 1, b_1 - 1)$; else if $s_1 = -2$ and if $(r_2, s_2) \in \{(2,0), (-1,-1), (-1,1)\}$, then $t_k = 30$ and $(a_1, b_1) \leftarrow (a_1 + 1, b_1 - 1)$.

Step 4: $SUM = t_k t_{k-1} ... t_1$ (the concatenation).

3. The discrete Fourier transform on the Pyxis structure

To consider the discrete Fourier transform (DFT) on the Pyxis structure, we need the concepts of abelian groups, and quotient groups.

Definition 2 An *abelian group* is a set G together with an associative binary operation + defined on G with the following properties:

- (Existence of a zero) There exists an element $0 \in G$ such that, for each $x \in G$, $x + 0 = x = 0 + x$.
- (Existence of an inverse) For each $x \in G$, there exists an element of G, denoted $-x$, such that $x + (-x) = 0 = (-x) + x$.
- (Commutativity) For any $x, y \in G$, we have $x + y = y + x$.

In this paper, all groups will be abelian and we write $x + (-y)$ as $x - y$ for any two elements $x, y \in G$. Let G_0 be a nonempty subset of a group G. If G_0 also forms a group under the operation + of G, then G_0 is called a *subgroup* of G. Now assume that G_0 is a subgroup of a group G. For any $p, q \in G$, if $p - q \in G_0$, then we say that p is congruent to q modulo G_0, denoted by $p \sim q$. For any $p \in G$, let $\overline{p} = \{u \in G \mid u \sim p\}$. Obviously $\overline{p} = p + G_0$ where $p + G_0$ denotes the set $\{u + y \in G \mid y \in G_0\}$. The set $\overline{p}$ is called a coset of G_0 in G. For any pair $p, q \in G$, it is easy to show that either $\overline{p} = \overline{q}$ or $\overline{p} \cap \overline{q} = \phi$. Define $\overline{p} + \overline{q} = \overline{p+q}$ and let $G / G_0 = \{p \mid p \in G\}$. It is easy to check that the set G / G_0 together with the binary operation + defined on G / G_0 is an abelian group. This group is called the *quotient group* of the group G by the subgroup G_0. By choosing one representative from each coset of G_0 in G, we obtain *a set of coset representatives* of the quotient group G / G_0.

Let L_0 be a nonempty subset of a lattice L. If L itself is a lattice, then L is called a *sublattice* of L. Obviously a lattice is an abelian group and a sublattice is a subgroup. For any $\phi \neq T \subseteq L$ and $x \in L$, to avoid confusion in some expressions, we let $T_x = T + x$. If T is a set of coset representatives of the quotient group L / L_0, then T *tiles the lattice* L *by translations by* *the sublattice* L_0 in the sense that $\bigcup\{T_p \mid p \in L_0\} = L$ and $T_p = T_q$ whenever $T_p \bigcap T_q = \phi$. Hence T is called a *tile* of L. Each tiling (tile) involved in this paper is a tiling (tile) by translations by a sublattice, and we just consider those sublattices of L which have the same dimension as L.

In the following, $\langle r,s \rangle$ denotes the inner product of $r,s \in R^2$. The *dual of a lattice* L is $L^* = \{s \in R^2 \mid \langle r,s \rangle \in Z, \forall r \in L\}$. Also, for any set G, let C^G denote the set of complex functions from a set G to the set C, and let $|G|$ denote the size of G. Now we are ready to give the definition of the DFT. Some of its properties can be found in [Zapata and Ritter, 2000].

Definition 3 Let L_0 be a sublattice of L, $G = L/L_0$, and $\hat{G} = L_0^{\,*}/L^*$. If P is a set of coset representatives of the quotient group L/L_0 and Q a set of coset representatives of $L_0^{\,*}/L^*$, then the *discrete Fourier transform* on P is the function $F: C^P \to C^Q$ defined by $F(a)(q) := \sum_{p \in G} a(p) \cdot e^{-2\pi i \langle p,q \rangle}$.

The following Theorem whose proof can be found in [Zheng,2007] shows that, for $2 < n \in N$, the n^{th} level of the Pyxis structure $P(n)$ does not tile the underlying lattice L_n. Hence its DFT cannot be defined using Definition 3. However we can add some lattice points to $P(n)$ as shown in Figure 7 to obtain a corresponding level of regular hexagonal structure (RHS) which was defined in [Vince and Zheng, 2007] and the DFT on the RHS can be computed efficiently. In Figure 7, the black hexagons are cells of $P(4)$, and the dashed green hexagons are the cells of the corresponding level of the RHS.

Theorem 6 For any $2 < n \in N$, the n^{th} level of the Pyxis structure $P(n)$ does not tile L_n.

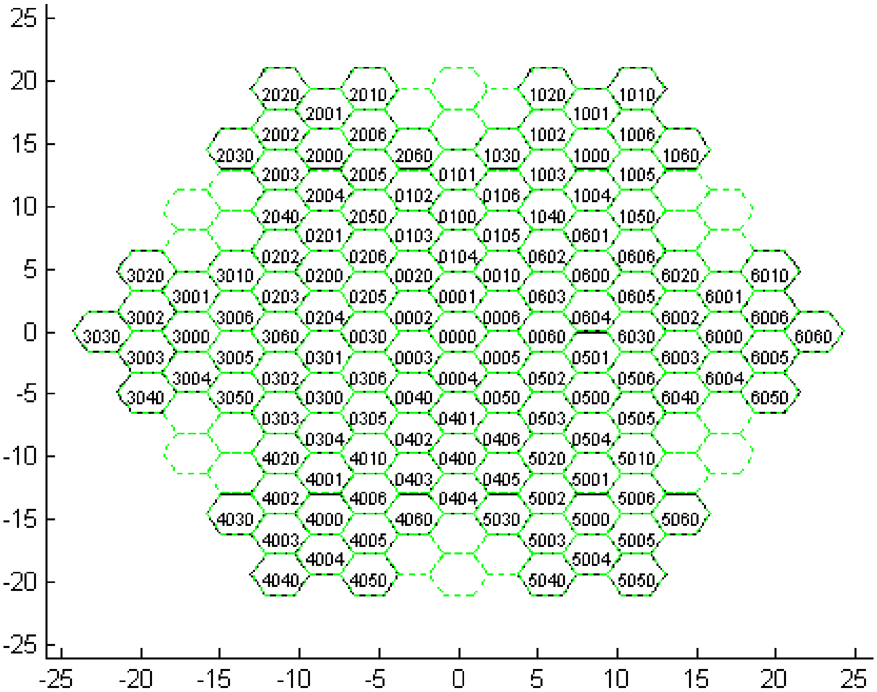

Figure 7 - Add some lattice points to P(4) to obtain the corresponding level of the RHS

4. Summary of this research

We have provided algebraic definition of the Pyxis structure and apply this definition to develop algorithms to add the labels of the Pyxis structure. The computational complexity of adding two labels of $P(n)$ is of linear order. The DFT on $P(n)$ is also discussed. It is shown that, for any $2 < n \in N$, the DFT cannot be defined on $P(n)$ because $P(n)$ does not tile the underlying lattice. However, as shown in Figure 5, $P(n)$ models one of the 32 arrays constituting the set of the sampled points of the sphere. We can add some lattice points as shown in Figure 7 to a given array to obtain the corresponding level of the regular hexagonal structure whose DFT can be computed efficiently.

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